F. anal. 6.1.2025

- Info, wiki, slides

0. Intro (Slides)

Fourier series and transforms

Math. analysis:

- conv. (difficult): pt.w, uniform, L^2
- properties, decay $\Leftrightarrow$ regularity

Applications:

- Weierstrass approx. thm.
- Weil equidistribution thm.
- Poisson summation thm.
- Heisenberg uncertainty principle

⋮

A. Intro Fourier series

Func'n $f : [a,b] \longrightarrow \mathbb{R}$ or $\mathbb{C}$, $L = b-a$

Def. 1:

F. coefficients:

$$a_n = \hat{f}(n) = \frac{1}{L}\int_a^b f(x)\, e^{-\frac{2\pi i n x}{L}}\, dx, \quad n \in \mathbb{Z}$$

F. series:

$$f(x) \sim S(f)(x) := \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{\frac{2\pi i n x}{L}}$$

(Symm.) partial sums:

$$S_N(f)(x) := \sum_{n=-N}^{N} \hat{f}(n)\, e^{\frac{2\pi i n x}{L}}, \quad N \in \mathbb{N}$$

Recall: Mat 4K:

a) $e^{ix} = \cos x + i \sin x$, $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$, ...

b) g L-periodic $\Rightarrow \int_0^L g = \int_a^{a+L} g$

$$\Rightarrow \quad \hat{f}(n) = \frac{1}{L}\int_{x_0}^{x_0+L} \cdots \text{ for any } x_0 \in \mathbb{R}$$

if f L-per.

c) Convergence:

$$S(f)(x) = \text{"}\lim_{N\to\infty}\text{"}\ S_N(f)(x)$$

Obs: Symm. sum! In general, sum $= \lim_{N\to\infty} \sum_{n=0}^{N} + \lim_{M\to\infty} \sum_{n=-M}^{-1}$

Problem 1: In what sense does $S_N(f) \underset{N\to\infty}{\longrightarrow} f$?

Ex. 1: $f(\theta) = \theta \,, \quad -\pi \le \theta \le \pi \qquad (L = 2\pi)$

$$\hat{f}(n) \overset{n\neq 0}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta e^{-in\theta}\, d\theta$$

$$\overset{\text{i.b.p.}}{=} \frac{1}{2\pi}\left[\frac{\theta}{-in} e^{-in\theta}\right]_{-\pi}^{\pi} + \frac{1}{2\pi i n}\int_{-\pi}^{\pi} e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\left(-\frac{\pi}{in}\right)\underbrace{\left(e^{-in\pi} + e^{in\pi}\right)}_{= 2\cos n\pi = 2(-1)^n} + \frac{1}{2\pi i n}\left(\frac{1}{-in}\right)\underbrace{\left(e^{-in\pi} - e^{in\pi}\right)}_{= -2i\sin n\pi = 0}$$

$$= \frac{(-1)^{n+1}}{in}$$

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta\, d\theta \overset{\text{odd}}{=} 0$$

$$f(\theta) \sim \sum_{n\neq 0} \frac{(-1)^{n+1}}{in} e^{in\theta} \underset{\sin nx = \dots}{\overset{\text{Chk.}}{=}} 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin n\theta$$

"Alternatating" series, conditionally conv., hard to see sum $= f$.

Ex. 2: $0 \le \theta \le 2\pi$

$$\frac{1}{4}(\pi - \theta)^2 \underset{2\times\text{i.b.p.}}{\sim} \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos n\theta \qquad (\text{abs. conv., sum} = ?)$$

$$\frac{\pi}{\sin \pi\alpha} e^{i(\pi-\theta)\alpha} \underset{\alpha\notin\mathbb{Z}}{\sim} \sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} e^{in\theta}$$

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Ex. 3: $\cancel{-\pi} \le x \le \pi$

$$\delta(x) \sim \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} 1 \cdot e^{inx} \qquad \text{(div. pt. w.)}$$

$$\uparrow$$

$$\int_0^{2\pi} \delta(x)\, e^{-inx}\, dx \overset{\text{def } \delta}{=} e^{-in0} = 1$$

Def. 2: Dirichlet kernel:

$$D_N(x) := 2\pi\, S_N(\delta)(x) = \sum_{n=-N}^{N} e^{inx}.$$

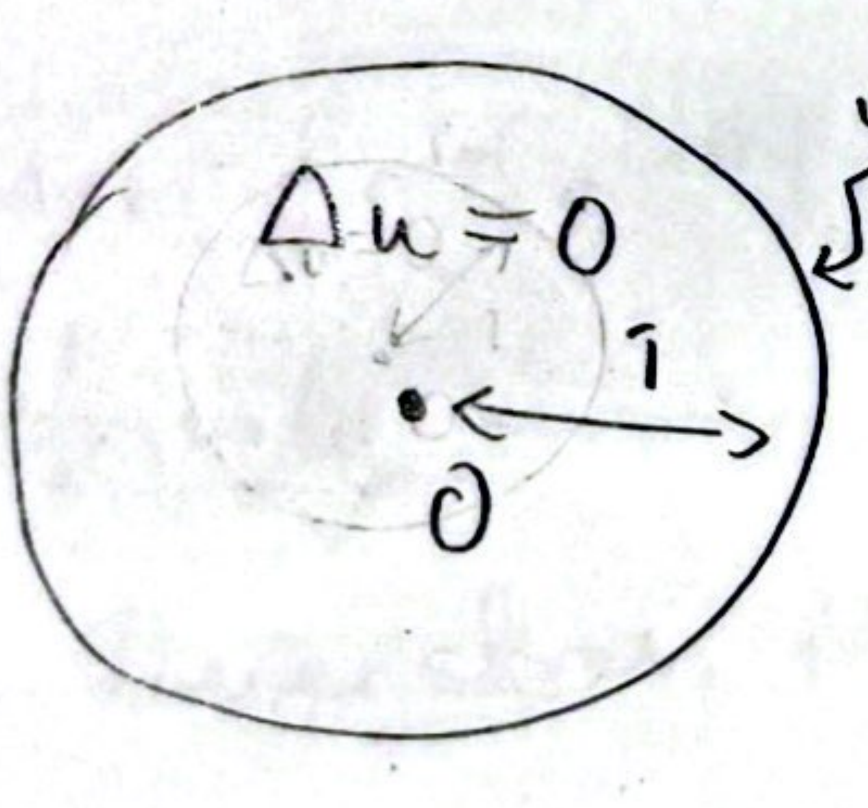
Obs 1:

$$D_N(x) \overset{w=e^{ix}}{=} \sum_{n=0}^{N} w^n + \sum_{n=-N}^{-1} w^n = \frac{1-w^{N+1}}{1-w} + w^{-N} \cdot \frac{1-w^N}{1-w}$$

$$= \frac{w^{-N} - w^{N+1}}{1-w} \cdot \frac{w^{-\frac{1}{2}}}{w^{-\frac{1}{2}}} = \frac{w^{-N-\frac{1}{2}} - w^{N+\frac{1}{2}}}{w^{-\frac{1}{2}} - w^{\frac{1}{2}}}$$

$$\overset{w=e^{ix}}{=} \underline{\underline{\frac{\sin(N+\frac{1}{2})x}{\sin\frac{1}{2}x}}}.$$

Ex. 4: Laplace eq'n in unit disk D_1 (SS sec. 1.2.1)



In polar coord.:

$u = u(r, \theta)$

$$\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0 & \text{for } r<1 \\ u = f(\theta) & \text{for } r = 1 \end{cases}$$

Separation of var's:

$$u(r,\theta) = F(r)\cdot G(\theta) \overset{\text{chk.}}{\Longrightarrow} \begin{cases} G''(\theta) + \lambda G(\theta) = 0 \\ r^2 F''(r) + r F'(r) - \lambda F(r) = 0 \end{cases}$$

Want G per. $\Rightarrow$ $\lambda = m^2 \geq 0$ and

$$G(\theta) = \tilde{A} e^{im\theta} + \tilde{B} e^{-im\theta}$$

Solve for F when $\lambda = m^2$:

$$F(r) \overset{\text{see book}}{=} A' r^m + B' \underbrace{r^{-m}}_{\text{unbnd/bnd in } D_1 \text{ if } m>0 / m\leq 0}$$

Want bnd. F (and u), so

$$u_m(r,\theta) = F_m G_m = r^{|m|} e^{im\theta}, \quad m \in \mathbb{Z},$$

and

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} a_m r^{|m|} e^{im\theta}, \quad r \in [0,1), \theta \in [-\pi,\pi]$$

$$\text{B.C.: } u(1,\theta) = \sum_{m=-\infty}^{\infty} a_m e^{im\theta} = f(\theta)$$

Def 3: Poisson kernel:

$$P_r(\theta) := \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}, \quad r \in [0,1), \theta \in [-\pi,\pi]$$

Obs 2: $P_r(\theta) = u(r,\theta)$ in Ex. 4 when $f(\theta) = \frac{1}{2\pi}\delta(\theta)$.

(i) Since $|r^{|n|} e^{in\theta}| \leq r^{|n|} < 1$, $P_r(\theta)$ conv. abs. and uniformly on $[0, r_0] \times [-\pi,\pi]$ for every $r_0 \in (0,1)$, by Weierstrass M-test (next time).

(ii) $\widehat{P_r}(n) = r^{|n|}$ $\left[\hat{P}(n) = \frac{1}{2\pi}\int \sum_m (\,) e^{-in\theta} d\theta \overset{\text{unif. conv.}}{=} \underbrace{\sum_m \frac{1}{2\pi}\int (\,) e^{-in\theta} d\theta}_{\text{next time}} \overset{\text{chk}}{=} r^{|n|} \right]$

$= 0$ if $m \neq n$

(iii) $P_r(\theta) = \sum_{n=0}^{\infty} w^n + \sum_{n=1}^{\infty} \overline{w}^n, \quad w = re^{i\theta}$

$$\overset{|w|<1}{\underset{\text{geom. series}}{=}} \frac{1}{1-w} + \overline{w}\frac{1}{1-\overline{w}} = \frac{1-\overline{w}+(1-w)\overline{w}}{(1-w)(1-\overline{w})}$$

$$= \frac{1-|w|^2}{|1-w|^2} \overset{\text{chk}}{=} \frac{1-r^2}{1-2r\cos\theta + r^2}$$

D_N and P_r appear in conv. pf's for F-series!

<u>Rem. 1:</u> $S_N(f)(x) \to f(x)$ pt. wise?

a) Not always true: Can change f in isolated pt's without changing $\hat{f}(n)$ and $S_N(f)(x)$.

b) There exists a cont. func'n that div. at a pt. (Du Bois-Raymond)

c) More regularity of $f \Rightarrow$ pt. wise (and maybe unif.) conv. (e.g. (piecewise) C^1 (Math. 4K))

d) Other limits require less regularity:

1) Cesaro and Abel means

2) Mean square L^2 conv.

e) Carleson (1966): $f \in L^2 \Rightarrow S_N(f)(x) \to f(x)$ pt. wise except on a set of measure 0

10:05 (!)

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Preliminary results

A. Continuity, uniform conv., and consequences

$f, f_n : U \longrightarrow \mathbb{R}$ (or $\mathbb{C}$), $U \subset \mathbb{R}^d$

Def. 1: $f_n \to f$ unif. in U if

$$\forall \varepsilon > 0 \ \exists N > 0 \text{ s.t. } |f_n(x) - f(x)| < \varepsilon \ \forall x \in U, \forall n > N$$

Sup-norm ("L^∞"-norm):

$$\| f \|_\infty := \sup_{x \in U} |f(x)|$$

Obs. 1: unif. conv. $\Longleftrightarrow$ conv. in $\|\cdot\|_\infty$ $\underset{\not\Leftarrow}{\Rightarrow}$ pt.-w. conv. in U

Continuous func's on $U \subseteq \mathbb{R}^d$: $C(U)$.

Lem. 1: (Lin. meth.) $U = K \subset \mathbb{R}^d$, closed, bnd (= compact)

a) $f \in C(K) \Rightarrow \| f \|_\infty = \max_{x \in K} |f(x)| < \infty$ (max attained on closed bnd. set)

b) $f \in C(K) \Rightarrow f$ bnd., $|f(x)| \leq \|f\|_\infty$ (from a))

c) $f_n \in C(K)$, $f_n \xrightarrow{\text{unif.}} f \Rightarrow f \in C(K)$

d) $(C(K), \|\cdot\|_\infty)$ is a Banach sp.

complete normed sp.

<u>Def. 2:</u> $f : U \longrightarrow \mathbb{R}$ (or $\mathbb{C}$)

(i) f <u>unif. cont.</u> if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x, y \in U$,

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

(ii) <u>Modulus of cont.</u> of f, $\omega_f : [0,\infty) \to [0,\infty]$,

$$\omega_f(r) := \sup\{ |f(x) - f(y)| : x, y \in U,\ |x-y| \le r\}$$

<u>Obs. 2:</u>

(i) cont. at x: $\delta = \delta(x,\varepsilon)$.

unif. cont.: $\delta = \delta(\varepsilon)$, good for <u>all</u> x

(ii) ω_f non-decreasing; f unif. cont. $\Leftrightarrow \omega_f(r) \xrightarrow[r\to 0^+]{} 0$

(iii) $|f(x) - f(y)| \le \omega_f(|x-y|)$

<u>Lem. 2:</u> $f \in C(K)$, $K \subset \mathbb{R}^d$ closed bnd. (= compact)

$\Rightarrow f$ is unif. cont. on K

<u>Pf:</u> ~~Lin. meth (?)~~. Exercise?

"<u>Pf.</u>": $\forall \varepsilon > 0, x \in K\ \exists \delta_x$ s.t. $|x-y| < \delta_x \Rightarrow |f(x) - f(y)| < \varepsilon$ (f cont.)

$K \subset \bigcup_{x\in K} B(x, \delta_x)$ cover; compact $\Rightarrow K \subset \bigcup_{i=1}^{N} B(x_i, \delta_{x_i})$ (sub cover)

Take $\delta < \min_{i=1,\dots,N} \delta_{x_i}$ so small that

$|x-y| < \delta \Rightarrow x, y \in B(x_j, \delta_{x_j})$ for some $j = j(x,y)$

$\Rightarrow |f(x) - f(y)| < \varepsilon$ □

Interchanging limits and integrals:

<u>Lem. 3</u>: If $C([a,b]) \ni f_n \underset{n\to\infty}{\overset{\text{unif.}}{\longrightarrow}} f$, then ($f \in C([a,b])$ and)

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

<u>Pf.</u>:

$$\left|\int_a^b f_n - \int_a^b f\right| \le \int_a^b |f_n - f|$$

$$\le (b-a)\,\|f_n - f\|_\infty \underset{n\to\infty}{\longrightarrow} 0 \qquad \square$$

<u>Cor. 3</u>: (Interchanging deriv. and integrals) If $f, \frac{\partial f}{\partial t} \in C(I \times [a,b])$, $t \ni I := [c,d]$ (closed), then

$$\frac{d}{dt}\int_a^b f(t,x)\,dx = \int_a^b \frac{\partial f}{\partial t}(t,x)\,dx$$

<u>Pf.</u>: $D_h^+ \varphi(t) := \frac{1}{h}\left(\varphi(t+h) - \varphi(t)\right) \overset{\text{fund. thm}}{\underset{\text{calc.}}{=}} \int_0^1 \varphi'(t+sh)\,ds$.

Since rectangle $I \times [a,b]$ closed, bnd. in $\mathbb{R}^2$, $\frac{\partial f}{\partial t}$ is unif. cont. (and bnd.). Hence $D_h^+ f \underset{h\to 0}{\overset{\text{unif}}{\longrightarrow}} \frac{\partial f}{\partial t}$:

$$\left|D_h^+ f(t,x) - \frac{\partial f}{\partial t}(t,x)\right| = \left|\int_0^1 \left(\frac{\partial f}{\partial t}(t+sh,x) - \frac{\partial f}{\partial t}(t,x)\right) ds\right|$$

$$\le \int_0^1 |\cdots|\,ds \le \int_0^1 \omega_{\frac{\partial f}{\partial t}}(sh)\,ds \le \omega_{\frac{\partial f}{\partial t}}(h).$$

Conclude by Lem. 3: $\left(\|D_h^+ f - \frac{\partial f}{\partial t}\|_\infty \le \omega_{\frac{\partial f}{\partial t}}(h)\right)$

$$\frac{d}{dt}\int f = \lim_{h\to 0} D_h^+ \int f \overset{\text{int. lin.}}{=} \lim_{h\to 0} \int D_h^+ f \overset{\text{Lem. 3}}{=} \int \frac{\partial f}{\partial t} \qquad \square$$

Cor. 2: (Termwise integr. of series), If $f_n \in C([a,b])$, $\sum_{n=-\infty}^{\infty} f_n(x)$ conv. unif. on $[a,b]$,

then
$$\int_a^b \sum_{n=-\infty}^{\infty} f_n(x)\,dx = \sum_{n=-\infty}^{\infty} \int_a^b f_n(x)\,dx.$$

Pf.:

$$\cancel{\int S_N\,dx = \int \sum_{-N}^{N} f_n\,dx = \sum_{-N}^{N} \int f_n\,dx}$$

$$\sum f_n \text{ conv. unif.} \iff \sum_{n=-N}^{N} f_n \xrightarrow[N\to\infty]{\text{unif}} \sum_{n=-\infty}^{\infty} f_n$$

Hence by Lem 3:

$$\int \sum_{n=-\infty}^{\infty} f_n = \int \lim \sum_{-N}^{N} f_n \overset{\text{Lem. 3}}{=} \lim \int \sum_{-N}^{N} f_n$$

$$\overset{\text{int. lin.}}{=} \lim \sum_{-N}^{N} \int f_n\,dx \overset{\text{def.}}{=} \sum_{n=-\infty}^{\infty} \int f_n\,dx \qquad \square$$

Unif. conv. of series:

Lem. 4: Weierstrass M-test

$$|f_n(x)| \le M_n \;\forall x \in [a,b], \quad \sum_{n=-\infty}^{\infty} M_n < \infty$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} f_n(x) \text{ conv. abs. and unif. for } x \in [a,b]$$

Pf.:

$$\Big\| \sum_{-\infty}^{\infty} f_n - \sum_{-N}^{N} f_n \Big\|_\infty \overset{\Delta\text{-ineq.}}{\le} \sum_{|n|>N} \|f_n\|_\infty \overset{\text{by assumption}}{\le} \sum_{|n|>N} M_n \xrightarrow[N\to\infty]{} 0 \qquad \square$$

Ex. 1: $\sum_{-\infty}^{\infty} |a_n| < \infty \Rightarrow \sum_{-\infty}^{\infty} a_n e^{inx}$ conv. abs. and unif.

B. Integrals and integrable func'ns

Partition of $[a,b]$:

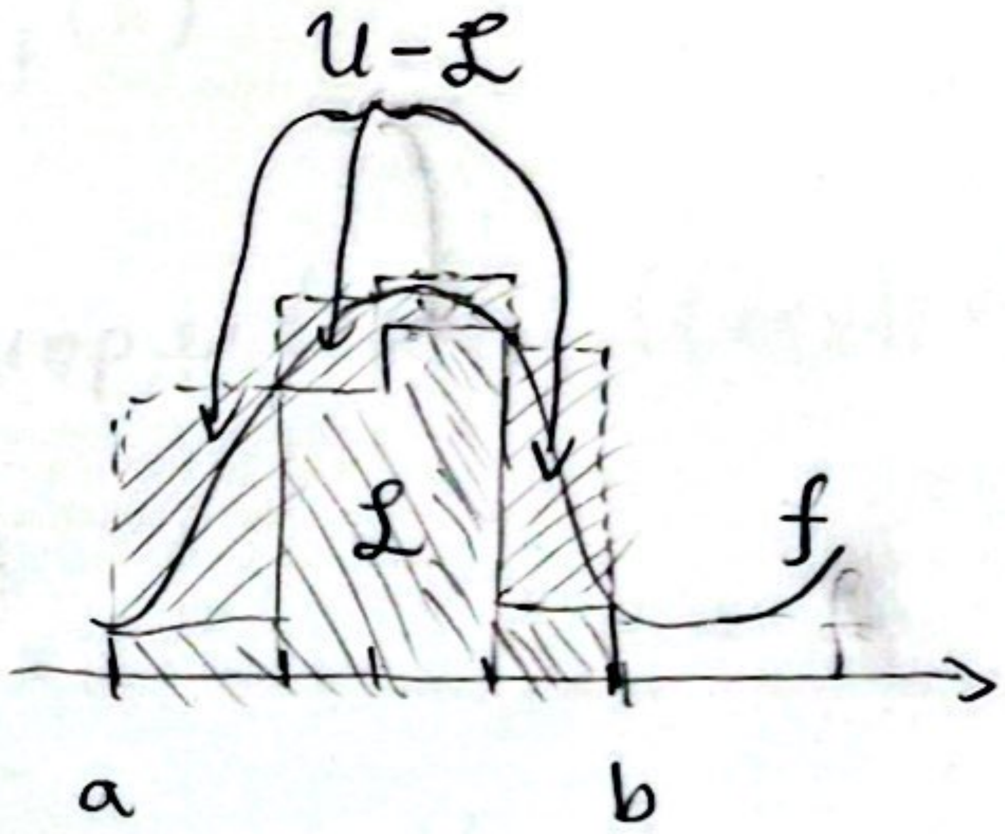
$$P : \quad a = x_0 < x_1 < \dots < x_N = b$$

$$I_j = [x_{j-1}, x_j], \quad |I_j| = x_j - x_{j-1}$$

Upper and lower sums:

$$\mathcal{U}(P,f) = \sum_{j=1}^{N} \sup_{x \in I_j} f(x) \cdot |I_j|$$

$$\mathcal{L}(P,f) = \sum_{j=1}^{N} \inf_{x \in I_j} f(x) \cdot |I_j|$$



<u>Def. 3:</u> $f : [a,b] \to \mathbb{R}$ <u>Riemann integrable</u> if

(i) f is bnd

(ii) $\forall \varepsilon > 0 \ \exists P$ s.t. $\mathcal{U}(P,f) - \mathcal{L}(P,f) < \varepsilon$.

<u>Obs. 3:</u> (i) If $P_1, P_2 \subset P'$ (P' refinement of $P_{1,2}$), then

$$\mathcal{L}(P_1, f) \le \mathcal{L}(P', f) \le \mathcal{U}(P', f) \le \mathcal{U}(P_2, f)$$

$$\left[\inf_{I_j \cup I_{j+1}} f \le \inf_{I_j} f, \inf_{I_{j+1}} f, \quad \sup_{I_j} f, \sup_{I_{j+1}} f \le \sup_{I_j \cup I_{j+1}} f \right]$$

<u>Lem. 4:</u> $f : [a,b] \to \mathbb{R}$ bnd.

(a) f bnd. $\Rightarrow$ $U := \inf_P \mathcal{U}(P,f)$, $L := \sup_P \mathcal{L}(P,f)$ exist and $U \le L$

(b) f R.-integrable $\Rightarrow$ $U = L =: I$, the <u>R. integral</u> of f

Pf.:

(a) $\mathcal{U}$ non-decreasing in P, bnd. below $\Rightarrow$ inf/lim $\exists$ by completeness of $\mathbb{R}$

(b) follows from (a) and Def. 3. $\square$

Lem. 5: $f \in C([a,b]) \Rightarrow f$ R. integrable

Pf.:

P: $a, a+h, \dots, a+(n-1)h, b$, $\quad h = \frac{b-a}{n}$

$$\mathcal{U}(P,f) - \mathcal{L}(P,f) = \sum |I_j| \underbrace{(\sup_{I_j} f - \inf_{I_j} f)}_{\overset{\text{chk.}}{\leq} \omega_f(h)} \leq (b-a)\,\omega_f(h) \underset{h\to 0}{\longrightarrow} 0$$

$\square$

12:02

Rem. 1: (SS Appendix)

(a) The R.-int. satisfies the usual properties ... $\int f+g = \int f + \int g$ ...

(b) f, g R.-int., $c \in \mathbb{R}$, $\varphi \in C(\mathbb{R})$

$\Rightarrow f+g,\ cf,\ f\cdot g$ and $\varphi(f)$ are all R.-int.

(c) f bnd., non-decreasing $\Rightarrow f$ R.-int.

(d) f piecewise cont. (PC) (or bnd. variation (BV)) $\Rightarrow f$ R. int.

[f BV $\Leftrightarrow f = g - h$, g, h as in (c). PC f'ns are BV]

(e) PC f'ns can have finite no. of jumps.

There are R-int. f'ns jumping on countable dense set in $[a,b]$

(f) $f = \chi_{\mathbb{Q}\cap[a,b]}(x)$ (rationals) $\Rightarrow \mathcal{U}(P,f) = 1$, $\mathcal{L}(P,f) = 0$, not R-int.

Thm. 1: (Approximation by cont. f'ns)

Let $\chi_{\mathcal{U}}(x) = \begin{cases} 1, & x \in \mathcal{U} \\ 0, & x \notin \mathcal{U} \end{cases}$, then

If f R.-int. on $[a,b]$, then $\exists \{f_k\}_{k=1}^{\infty} \subset C([a,b])$ s.t.

(i) $\|f_k\|_\infty \leq \|f\|_\infty$, and

(ii) $\int_a^b |f(x) - f_k(x)|\,dx \underset{k\to\infty}{\longrightarrow} 0.$